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GENERALIZATION OF RKM INTEGRATORS FOR SOLVING A CLASS OF EIGHTH-ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH APPLICATIONS

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Abstract. The main contribution of this work is the development of direct explicit methods of RK-type for solving class of eighth-order ordinary differential equations (ODEs) to improve the computational efficiency of the methods. For this purpose, we have generalized RK, RKN, RKD, RKT, RKFD and RKM methods for solving class of first-, second-, third-, fourth-, fifth-, sixth- and seventh-orders ODEs. Using Taylor-expansion approach, we have derived the algebraic equations of the order-conditions for the proposed RKM integrators. Based on these order-conditions, proposed method of eighth-order with five-stage has been derived. The zero stability of the method is proven. Stability polynomial of the method for linear special eighth-order ODE is given. Numerical results have clearly shown the advantage and the efficiency of the new method and agree well with analytical solutions due to the fact that the proposed integrators are zero stable, more efficient and accurate integrators.

Keywords: Runge-Kutta method (RK), RKM, RKT, RKN, Order-conditions, Taylor expansion.
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1 Introduction

Differential equations (DEs); ordinary or partial; are significant tools for mathematical models in applications of science, economics and engineering. High-order differential equation arises in many fields of science and engineering such as nonlinear optics and quantum mechanics. Eighth-order differential equations govern physics of some hydrodynamic stability problems (Wazwaz, 2002). The approximated and numerical solutions for ordinary differential equations (ODEs) are very important in scientific computation, as they are widely used to model real world problems. The derivation of solutions of the nonlinear problems by analytical techniques is often rather difficult. Various approximated, analytical and numerical methods have been proposed to solve such problems (Abbasbandy & Shirzadi, 2008). When this instability starts as an overst ability, it is modelled by an eighth-order ODE. Furthermore, Mechee & Kadhim (2016), Mechee & Kadhim (2016), Mechee (2019), Mechee & Mshachal (2019) and Reddy (2017) have derived numerical methods of RK-type for solving classes of ODEs of third-, fourth-, fifth-, sixth- and seventh-order respectively. However, to improve the computational efficiency of the indirect numerical methods for solving class of eighth-order ODEs, the indirect numerical methods improved to be direct, more efficient and accurate methods. Using Taylor expansion approach, we have derived the algebraic equations of the order-conditions for the proposed RKM integrators up to the tenth-order. Based on these order-conditions, RKM method of fifth- and eighth-order has been derived. The novelty of this work is the generalization RK, RKN, RKD, RKT, RKFD and RKM methods for solving the classes of order less then eighth-order ODEs. Numerical implementations using Maple and MATLAB show that the numerical solutions of proposed integrator agree well with analytical solution due to the fact the proposed integrator is efficient and accurate and has less computational time comparing with indirect methods. The are several applications of seventh-order ODEs such that KdV, which is a nonlinear PDE, and the another application is nonlinear dispersive equations, which include several models arising in the study of different physical phenomena. Ray & Gupta (2017) has studied an efficient numerical solution for seventh-order DEs while Faires & Burden (2003) has proposed method based on the Legendre wavelets to solve sKdV. In this paper, a class of eighth-order ODEs has been studied and direct explicit method of RK-type for solving this class has been derived.

2 Preliminary

Here, we give the following definition for general class of quasi linear eighth-order ODE as the follows:

2.1 Class of Eighth-Order Ordinary Differential Equations

In this study, we concerned with class of eighth-order ODEs with no appearance for the derivatives up to seventh-order. It can be written in the following form

$$y^{(8)}(t) \quad = \quad f(t, y(t)); \quad t \geq t_0 \tag{1}$$

subject to initial condition: $y(t_0) = \overline{\alpha_i}; i = 0, 1, \ldots, 7$.

where $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ and $y(t) = [y_1(t), y_2(t), \ldots, y_N(t)]$.

Knowing that, f is vector of independent variables of N components of the system of ODE (1). To convert the function $f(t, y(t))$ which depends on two variables t and $y(t)$ to a function which depends only on dependent variable $y(t)$, using high dimension we can work in N+1 dimension using the assumption $y_{N+1}(t) = t$, then Equation (1) can be simplified to Equation (2)

$$v^{(8)}(t) \quad = \quad h(v(t)) \tag{2}$$

Using the following consideration

$$v(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \cdots \\ \cdots \\ \cdots \\ y_N(t) \\ t, \end{pmatrix} \qquad h(v(t)) = \begin{pmatrix} f_1(v_1, v_2, \ldots, v_{N+1}) \\ f_2(v_1, w_2, \ldots, v_{N+1}) \\ f_3(v_1, v_2, \ldots, v_{N+1}) \\ \cdots \\ \cdots \\ \cdots \\ f_N(v_1, v_2, \ldots, v_{N+1}) \\ 0 \end{pmatrix}$$

subject to the initial condition $y^{'}(t_0) = \overline{\alpha_i}; i = 0, 1, \ldots, 7$ where $\overline{\alpha_i} = [\overline{\alpha}_1, \overline{\alpha}_2, \overline{\alpha}_3, \ldots, \overline{\alpha}_N, t_0]$

Some of engineers and scientists used to solve Equation (1) or (2) by multistep method (Lmm). Mostly, they used to solve ODEs of higher-order by converting them to equivalent system of first-order ODEs and they solved using a classical RK method (Jackiewicz et al., 2003). However, it would be more efficient if eighth-order ODE can be solved using proposed direct RKM method. In this paper, we are concerned with direct explicit RKM integrators for solving class of eighth-order ODEs. Using Taylor series expansion approach, we have obtained the order-conditions of the proposed methods. Consequently, we have derived two RKM integrators based on these algebraic order-conditions.

3 Proposed RKM Method

The formula of proposed explicit RKM integrator with s-stage for solving eighth-order ODE (1) written as following

$$y_{n+1} = y_n + hy_n^{'} + \frac{h^2}{2}y_n^{''} + \frac{h^3}{6}y_n^{'''} + \frac{h^4}{24}y_n^{''''} + \frac{h^5}{120}y_n^{'''''} + \frac{h^6}{720}y_n^{''''''} + \frac{h^7}{5040}y_n^{''''''} + h^8\sum_{i=1}^{s} b_i k_i, \quad (3)$$

$$y_{n+1}^{'} = y_n^{'} + hy_n^{''} + \tfrac{h^2}{2}y_n^{'''} + \tfrac{h^3}{6}y_n^{''''} + \tfrac{h^4}{24}y_n^{'''''} + \tfrac{h^5}{120}y_n^{''''''} + \tfrac{h^6}{720}y_n^{''''''} + h^7 \textstyle\sum_{i=1}^{s} b_i^{'} k_i, \quad (4)$$

$$y_{n+1}^{''} = y_n^{''} + hy_n^{'''} + \frac{h^2}{2}y_n^{''''} + \frac{h^3}{6}y_n^{'''''} + \frac{h^4}{12}y_n^{''''''} + \frac{h^5}{120}y_n^{''''''} + h^6\sum_{i=1}^{s} b_i^{''} k_i, \quad (5)$$

$$y_{n+1}^{'''} = y_n^{'''} + hy_n^{''''} + \frac{h^2}{2}y_n^{'''''} + \frac{h^3}{6}y_n^{''''''} + \frac{h^4}{12}y_n^{''''''} + h^5\sum_{i=1}^{s} b_i^{'''} k_i, \quad (6)$$

$$y_{n+1}^{''''} = y_n^{''''} + hy_n^{'''''} + \frac{h^2}{2}y_n^{''''''} + \frac{h^3}{6}y_n^{''''''} + h^4\sum_{i=1}^{s} b_i^{''''} k_i, \quad (7)$$

$$y_{n+1}^{'''''} = y_n^{'''''} + hy_n^{''''''} + \frac{h^2}{2}y_n^{''''''} + h^3\sum_{i=1}^{s} b_i^{'''''} k_i, \quad (8)$$

$$y_{n+1}^{''''''} = y_n^{''''''} + hy_n^{''''''} + h^2\sum_{i=1}^{s} b_i^{''''''} k_i, \quad (9)$$

$$y_{n+1}^{'''''''} = y_n^{'''''''} + h\sum_{i=1}^{s} b_i^{'''''''} k_i, \quad (10)$$

and

$$\begin{aligned} k_1 &= f(t_n, y_n), & (11)\\ k_i &= f(t_n + c_i h, y_n + hc_i y_n^{'} + \frac{h^2}{2}c_i^2 y_n^{''} + \frac{h^3}{6}c_i^3 y_n^{'''} + \frac{h^4}{24}c_i^4 y_n^{''''} + \frac{h^5}{120}c_i^5 y_n^{'''''} & \\ &+ \frac{h^6}{720}c_i^6 y_n^{''''''} + \frac{h^7}{5040}c_i^7 y_n^{''''''} + h^8\sum_{j=1}^{i-1} a_{ij}k_j). & (12) \end{aligned}$$

The parameters of RKM integrator are $a_{ij}, c_i, b_i, b_i^{'}, b_i^{''}, b_i^{'''}, b_i^{''''}, b_i^{'''''}, b_i^{''''''}, b_i^{''''''}$ and $b_i^{'''''''}$ for $i, j = 1, 2, \cdots, s$ are real and h is the step-size. RKM is an explicit integrator if $a_{ij} = 0$ for $i < j$ and otherwise RKM is implicit integrator. The coefficients of RKM method have been expressed in Butcher table as in Table 1 The order-conditions of RKM integrators for solving seventh-order

$$\begin{array}{c|c} C & A \\ \hline & b^T \\ & b'^T \\ & b''^T \\ & b'''^T \\ & b''''^T \\ & b'''''^T \\ & b''''''^T \\ & b'''''''^T \end{array}$$

Table 1: Butcher Table of RKM method

ODEs have been derived by Mechee et al. (2016). In this study, using the same technique, we have derived the algebraic equations of order-conditions of RKM methods for solving class of eighth-order ODEs.

3.1 The Order-Conditions Derivation of RKM Methods

The order-conditions of RKM integrators can be obtained from the direct expansion of the local truncation error. RKM formulae in (3)-(12) can be expressed as follows $y_{n+1}^{(i)} = y_n + h\omega^{(i)}(t_n, y_n); \quad i = 0, 1, \ldots, 6$ where the increment functions are defined as the following

$$\omega(t_n, y_n) = y_n^{(1)} + \frac{h}{2}y_n^{''} + \frac{h^2}{6}y_n^{'''} + \frac{h^3}{24}w_n^{''''} + \frac{h^4}{120}y_n^{'''''} + \frac{h^5}{720}y_n^{''''''} + \frac{h^6}{5040}y_n^{'''''''} + h^7\sum_{i=1}^{s} b_i k_i,$$

$$\omega^{(1)}(t_n, y_n) = y_n^{(2)} + \frac{h}{2}y_n^{'''} + \frac{h^2}{6}y_n^{''''} + \frac{h^3}{24}y_n^{'''''} + \frac{h^4}{120}y_n^{''''''} + \frac{h^5}{720}y_n^{'''''''} + h^6\sum_{i=1}^{s} b_i^{'} k_i^{'},$$

$$\omega^{(2)}(t_n, y_n) = y_n^{(3)} + \frac{h}{2}y_n^{''''} + \frac{h^2}{6}y_n^{'''''} + \frac{h^3}{24}y_n^{''''''} + \frac{h^4}{120}y_n^{'''''''} + h^5\sum_{i=1}^{s} b_i^{''} k_i^{''},$$

$$\omega^{(3)}(t_n, y_n) = y_n^{(4)} + \frac{h}{2}y_n^{'''''} + \frac{h^2}{6}y_n^{'''''''} + \frac{h^3}{24}y_n^{''''''} + h^4\sum_{i=1}^{s} b_i^{'''} k_i^{'''},$$

$$\omega^{(4)}(t_n, y_n) = y_n^{(5)} + \frac{h}{2}y_n^{''''''} + \frac{h^2}{6}y_n^{'''''''} + h^3\sum_{i=1}^{s} b_i^{''''} k_i^{''''},$$

$$\omega^{(5)}(t_n, y_n) = y_n^{(6)} + \frac{h^2}{2}y_n^{'''''''} + h^2\sum_{i=1}^{s} b^{'''''} k_i^{'''''},$$

$$\omega^{(6)}(t_n, y_n) = y_n^{(7)} + h\sum_{i=1}^{s} b_i^{''''''} k_i^{''''''},$$

$$\omega^{(7)}(t_n, y_n) = \sum_{i=1}^{s} b_i^{''''''} k_i^{''''''},$$

where, k_i is defined as follow

$$k_1 = f(t_n, y_n), \tag{13}$$

$$\begin{aligned} k_i = f(t_n + c_i h, y_n + hc_i y_n^{'} + \frac{h^2}{2}c_i^2 y_n^{''} + \frac{h^3}{6}c_i^3 y_n^{'''} + \frac{h^4}{24}c_i^4 y_n^{''''} + \frac{h^5}{120}c_i^5 y_n^{'''''} \\ + \frac{h^6}{720}c_i^6 y_n^{''''''} + h^7\sum_{j=1}^{i-1} a_{ij} k_j), \end{aligned} \tag{14}$$

for $i = 2, 3, ..., s$. If Δ represents Taylor series increment function and the local truncation errors of the derivatives of the solution of order zero up to order seven can be obtained by substituting the analytical solution $y(t)$ of ODE (1) into the RKM increment function. This gives

$$\tau_{n+1}^{(i)} \quad = \quad h(\omega^{(i)} - \Delta^{(i)})$$

for $i = 1, 2, \ldots, 6$. These expressions have given in elementary differentials terms also Taylor series increment can be expressed as follows

$$\begin{aligned}
\Delta^{(0)} &= y' + \frac{h}{2}y'' + \frac{h^2}{6}y''' + \frac{h^3}{24}y'''' + \frac{h^4}{120}y''''' + \frac{h^5}{720}y'''''' + \frac{h^5}{5040}y''''''' + O(h^7),\\
\Delta^{(1)} &= y' + \frac{h}{2}y'' + \frac{h^2}{6}y''' + \frac{h^3}{24}y'''' + \frac{h^4}{120}y''''' + \frac{h^5}{720}y'''''' + O(h^6),\\
\Delta^{(2)} &= y'' + \frac{h}{2}y''' + \frac{h^2}{6}y'''' + \frac{h^3}{24}y''''' + \frac{h^4}{120}y'''''' + O(h^5),\\
\Delta^{(3)} &= y''' + \frac{h}{2}y'''' + \frac{h^2}{6}y''''' + \frac{h^3}{24}y'''''' + O(h^4),\\
\Delta^{(4)} &= y'''' + \frac{h}{2}y''''' + \frac{h^2}{6}y'''''' + O(h^3),\\
\Delta^{(5)} &= y''''' + \frac{h}{2}y'''''' + O(h^2),\\
\Delta^{(6)} &= y'''''' + O(h),\\
\Delta^{(7)} &= O(1).
\end{aligned}$$

3.2 Order-Conditions

To determine the order-conditions of the numerical integrators indicated by equations (3)-(12), RKM method formula is expanded using the approach of Taylor's series expansion using Maple software. Hence the following order-conditions are given

$$y : \sum b_i = \frac{1}{5040}, \sum b_i c_i = \frac{1}{40320}, \sum b_i c_i^2 = \frac{1}{181440}, \sum b_i c_i^3 = \frac{1}{604800}, \tag{15}$$

$$y' : \sum b_i' = \frac{1}{720}, \sum b_i' c_i = \frac{1}{5040}, \sum b_i' c_i^2 = \frac{1}{20160} \sum b_i' c_i^3 = \frac{1}{60480} \sum b_i' c_i^4 = \frac{1}{151200}. \tag{16}$$

$$y'' : \sum b_i'' = \frac{1}{720}, \sum b_i'' c_i = \frac{1}{5040}, \sum b_i'' c_i^2 = \frac{1}{20160}, \sum b_i'' c_i^3 = \frac{1}{60480}, \sum b_i'' c_i^4 = \frac{1}{151200},$$

$$. \sum b_i'' c_i^5 = \frac{1}{332640}. \tag{17}$$

$$y''' : \sum b_i''' = \tfrac{1}{120}, \sum b_i''' c_i = \tfrac{1}{720} \sum b_i''' c_i^2 = \tfrac{1}{2520} \sum b_i''' c_i^3 = \tfrac{1}{6720} \sum b_i''' c_i^4 = \tfrac{1}{15120}, \sum b_i''' c_i^5 = \tfrac{1}{30240},$$

$$. \sum b_i''' c_i^6 = \frac{1}{55440}. \tag{18}$$

$$y'''' : \sum b_i'''' = \tfrac{1}{24}, \sum b_i'''' c_i = \tfrac{1}{120}, \sum b_i'''' c_i^2 = \tfrac{1}{360}, \sum b_i'''' c_i^3 = \tfrac{1}{840}, \sum b_i'''' c_i^4 = \tfrac{1}{1680}, \sum b_i'''' c_i^5 = \tfrac{1}{3024},$$

$$. \sum b_i'''' c_i^6 = \frac{1}{5040}, \sum b_i'''' c_i^7 = \frac{1}{7920}. \tag{19}$$

$$y''''' : \sum b_i''''' = \tfrac{1}{6}, \sum b_i''''' c_i = \tfrac{1}{24}, \sum b_i''''' c_i^2 = \tfrac{1}{60} \sum b_i''''' c_i^3 = \tfrac{1}{120}, \sum b_i''''' c_i^4 = \tfrac{1}{210}, \sum b_i''''' c_i^5 = \tfrac{1}{336},$$

$$. \sum b_i''''' c_i^6 = \frac{1}{504}, \sum b_i''''' c_i^7 = \frac{1}{720}, \sum b_i''''' c_i^8 = \frac{1}{990}. \tag{20}$$

$$y'''''' : \sum b_i'''''' = \tfrac{1}{2}, \sum b_i'''''' c_i = \tfrac{1}{6}, \sum b_i'''''' c_i^2 = \tfrac{1}{12}, \sum b_i'''''' c_i^3 = \tfrac{1}{20}, \sum b_i'''''' c_i^4 = \tfrac{1}{30}, \sum b_i'''''' c_i^5 = \tfrac{1}{42},$$

$$.\sum b_i^{'''''} c_i^6 = \frac{1}{56}, \sum b_i^{'''''} c_i^7 = \frac{1}{72}, \sum b_i^{'''''} c_i^8 = \frac{1}{90}, \sum b_i^{'''''} c_i^9 = \frac{1}{110}, \tag{21}$$

$$y^{''''''} : \sum b_i^{''''''} = 1, \sum b_i^{''''''} c_i = \frac{1}{2}, \sum b_i^{''''''} c_i^2 = \frac{1}{3}, \sum b_i^{''''''} c_i^3 = \frac{1}{4}, \sum b_i^{''''''} c_i^4 = \frac{1}{5}, b_i^{''''''} c_i^5 = \frac{1}{6},$$

$$.\sum b_i^{''''''} c_i^6 = \frac{1}{7}, \sum b_i^{''''''} c_i^7 = \frac{1}{8}, \sum b_i^{''''''} c_i^8 = \frac{1}{9}, \sum b_i^{''''''} c_i^9 = \frac{1}{10}, \sum b_i^{''''''} c_i^{10} = \frac{1}{11},$$

$$.\sum_{i=2,j=1}^{5,4} \sum_{j<i} b_i^{'''''''} a_{ij} = \frac{1}{120}, \sum_{i=2,j=1}^{5,4} \sum_{j<i} b_i^{''''''} c_i a_{ij} = \frac{1}{144}, \sum_{i=3,j=2}^{5,4} \sum_{j<i} b_i^{''''''} a_{ij} c_j = \frac{1}{720},$$

$$.\sum_{i=2,j=1}^{5,4} \sum_{j<i} b_i^{''''''} a_{ij} c_i^2 = \frac{1}{2520}, \sum_{i=2,j=1}^{5,4} \sum_{j<i} b_i^{''''''} c_j^2 a_{ij} = \frac{1}{168}, \sum_{i=3,j=2}^{5,4} \sum_{j<i} b_i^{''''''} c_j a_{ij} c_j = \frac{1}{840}. \tag{22}$$

3.3 Derivation of RKM Methods

To derive proposed RKM methods using the algebraic conditions (15)-(22) with the following assumption $b_i = \frac{(1-c_i)^7}{7!} b_i^{'''''''}, b_i^{'} = \frac{(1-c_i)^6}{6!} b_i^{'''''''}, b_i^{''} = \frac{(1-c_i)^5}{5!} b_i^{'''''''}, b_i^{'''} = \frac{(1-c_i)^4}{4!} b_i^{'''''''}, b_i^{''''} = \frac{(1-c_i)^3}{3!} b_i^{'''''''}$, $b_i^{'''''} = \frac{(1-c_i)^2}{2!} b_i^{'''''''}, b_i^{''''''} = (1-c_i) b_i^{'''''''}$, for $i = 1, \ldots, s$,

The parameters of RKM method $c_i, a_{ij}, b_i, b_i^{'}, b_i^{''}, b_i^{'''}, b_i^{''''}, b_i^{'''''}$ and $b_i^{''''''}$ for $i, j = 1, 2, \ldots, s$. for Four-stage fifth-order and five-stage sixth-order RKM integrators have been evaluated and the Butcher tableaus of these integrators are shown in the Table 2.

0	0	0	0	0	0
$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0
1	$\frac{-4}{21} - \frac{10\sqrt{21}}{21}$	$\frac{727}{252} + \frac{10\sqrt{21}}{21}$	0	0	0
$\frac{1}{2} - \frac{\sqrt{21}}{14}$	$\frac{1}{2}$	$\frac{-449}{2744} + \frac{45\sqrt{21}}{392}$	$\frac{36}{343} - \frac{\sqrt{21}}{28}$	0	0
$\frac{1}{2} + \frac{\sqrt{21}}{14}$	$\frac{5}{686} - \frac{169\sqrt{21}}{1372}$	$\frac{-181}{2744} + \frac{555\sqrt{21}}{2744}$	0	$\frac{1}{2}$	0
	$\frac{1}{100800}$	$\frac{1}{1814400}$	0	$\frac{13}{1814400} + \frac{139\sqrt{21}}{88905600}$	$\frac{13}{1814400} - \frac{139\sqrt{21}}{88905600}$
	$\frac{1}{14400}$	$\frac{11}{129600}$	0	$\frac{11}{181440} + \frac{\sqrt{21}}{75600}$	$\frac{11}{181440} - \frac{\sqrt{21}}{75600}$
	$\frac{1}{2400}$	$\frac{1}{10800}$	0	$\frac{19}{43200} + \frac{29\sqrt{21}}{302400}$	$\frac{19}{43200} - \frac{29\sqrt{21}}{302400}$
	$\frac{1}{480}$	$\frac{1}{1080}$	0	$\frac{23}{8640} + \frac{\sqrt{21}}{1728}$	$\frac{23}{8640} - \frac{\sqrt{21}}{1728}$
	$\frac{1}{120}$	$\frac{1}{135}$	0	$\frac{7}{540} + \frac{\sqrt{21}}{360}$	$\frac{7}{540} - \frac{\sqrt{21}}{360}$
	$\frac{1}{40}$	$\frac{2}{45}$	0	$\frac{7}{144} + \frac{7\sqrt{21}}{144}$	$\frac{7}{144} - \frac{7\sqrt{21}}{144}$
	$\frac{1}{20}$	$\frac{8}{45}$	0	$\frac{49}{360} + \frac{7\sqrt{21}}{360}$	$\frac{49}{360} - \frac{7\sqrt{21}}{360}$
	$\frac{1}{20}$	$\frac{16}{45}$	$\frac{1}{20}$	$\frac{49}{180}$	$\frac{49}{180}$

Table 2: The Butcher Tableau for the RKM Method of Four Stages and Fifth Order.

4 Stability of the Method

4.1 Zero Stability of the Method

Definition 1. *The method is said to be zero stable if it satisfied $-1 < \xi \leq 1$. (Jackiewicz et al., 2003)*

Zero-stability is an important tool for proving the stability and convergence of linear multi-step methods. We can rewrite the equations (3)-(12) as follows

Thus the characteristic polynomial is

$$\rho(\xi) \quad = \quad (\xi - 1)^7. \tag{23}$$

Hence, the method is zero-stable since the roots are $\xi = 1, 1, 1, 1, 1, 1, 1$, are less or equal to one.

4.2 Absolute Stability of the Method

Definition 2. *The method is said to be absolutely stable for a given roots if all the roots lies within the unit circle (Jackiewicz et al., 2003).*

Mechee et al. (2016) has studied the absolute stability for RKD method and compared the stability regions for RKD and RKT methods, In the same way we have studied the absolute stability for the RKM method and we apply this to the test problem

$$w^{(8)} \quad = \quad -\chi^8 w. \tag{24}$$

Now, consider formulas Equations (3)-(12), which can be written for the test problem as follows

The stability function associated with this method is given by,

$$\Theta(\xi, H) = \mid \xi I - RH \mid,$$

where $R(H)$ a rational function of H, H is a stability matrix the characteristic equation of which can be written as

$$\Theta(\xi, H) = \lambda_0(H)\xi^8 + \lambda_1(H)\xi^7 + \lambda_2(H)\xi^6 + \lambda_3(H)\xi^5 + \lambda_4(H)\xi^4 + \lambda_5(H)\xi^3 + \lambda_6(H)\xi^2 + \lambda_7(H)\xi + \lambda_8(H).$$

5 Applications

In this section, we imply the proposed methods for solving four problems and their numerical results introduced in Figure 1

Example 1. *(Linear ODE)*

$$y^{(8)}(t) = y(t); \qquad 0 < t \leq 1.$$

subject to ICs: $y^{(i)}(0) = (-1)^i; i = 0, 1, \dots, 7.$

The exact solution is $y(t) = e^{-t}$

Example 2. *(Linear ODE)*

$$y^{(8)}(t) = 256y(t); \qquad 0 < t \leq 1.$$

subject to ICs: $y^{(i)}(0) = (-2)^i; i = 0, 1, \dots, 7.$

The exact solution is $y(t) = e^{-2t}$

Example 3. *(Non Constant Coefficients ODE)*

$$y^{(8)}(t) = (1680 - 13440t^2 + 13440t^4 - 3584t^6 + 256t^8)y(t); \qquad 0 < t \leq 1.$$

subject to ICs: $y^{(2i+1)}(0) = 0; i = 0, 1, 3, y(0) = 1, y''(0) = -2, y^{(4)}(0) = 12, y^{(6)}(0) = -120.$

The exact solution is $y(t) = e^{t^2}$

Example 4. *(Linear ODE)*

$$y^{(8)}(t) = y(t), \qquad 0 < t \leq 1.$$

subject to ICs: $y^{(2i+1)}(0) = 0$ *for* $i = 0, 1, 2, 3$ *and* $y^{(2i)}(0) = (-1)^i$ *for* $i = 0, 1, 2, 3$

The exact solution is $y(t) = cos(t)$.

Example 5. *(Non Linear ODE)*

$$y^{(8)}(t) = -\frac{5040}{(1+t)^8}, \qquad 0 < t \leq 1.$$

subject to ICs: $y(0) = 0, y'(0) = 1, y''(0) = -1, y'''(0) = 2, y^{(4)}(0) = -6, y^{(5)}(0) = 24, y^{(6)}(0) = -120, y^{(7)}(0) = 720.$

The exact solution is $y(t) = log(1 + t)$

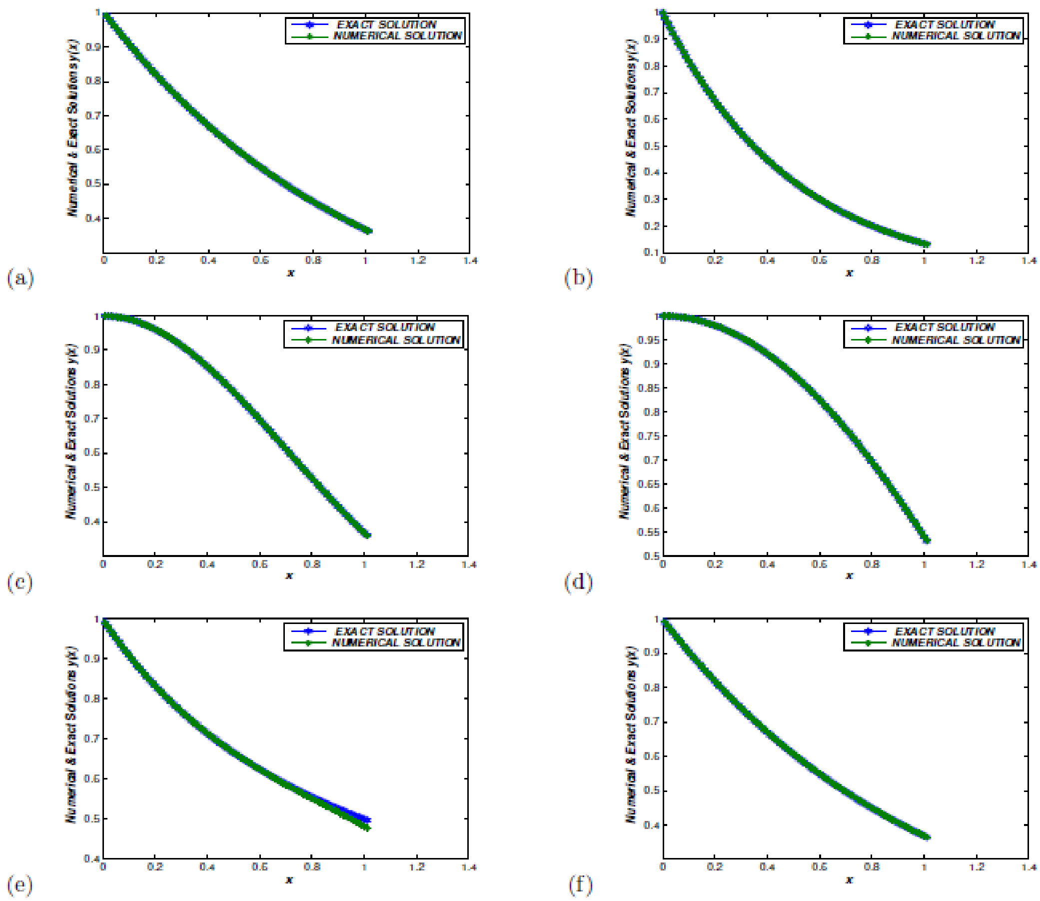

Figure 1: Comparisons on approximated solutions versus exact solutions for Problems (a) 1, (b) 2, (c) 3, (d) 4, (e) 5 and (f) 6

Example 6. *(Linear ODE with Relatively Long Interval)*

$$y^{(7)}(t) = 0.0000001e^{-\frac{t}{10}}, \qquad 0 < t \leq 1.$$

subject to ICs: $y(0) = 1, y^{(i)}(0) = (-0.1)^i; i = 1, 2, \ldots, 6.$

The exact solution is $y(t) = e^{-\frac{t}{10}}$

6 Conclusion

In this study, we have derived the algebraic equations of order-conditions for direct integrators of RKM for class of eighth-order ODEs. The approach of the derivation of the proposd method is based on Taylor expansion. The objective of this work is to establish direct explicit integrators of RK type for solving class of eighth-order ODEs. For this purpose, we have generalized the integrators RK, RKN, RKD, RKT and RKFD which are used for solving class of order less than seventh-order. We have derived eighth-order with five-stage RKM method. Numerical results using the proposed method have been compared with analytical solutions in figures 1-6 which show that the numerical results of the six test problems are more efficient and accurate as well-known existing methods. The new integrates are more efficient in implementation as they require less function evaluations.

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References

Abbasbandy, S., Shirzadi, A. (2008). The variational iteration method for a class of eighth-order boundary-value differential equations. *Zeitschrift für Naturforschung A, 63*(12), 745-751.

Faires, J.D., Burden, R. (2003). *Numerical Methods*, Thomson Learning. Inc., Pacific Grove.

Jackiewicz, Z., Renaut, R. & Feldstein, A. (1991). Two-step Runge-Kutta methods. *SIAM Journal on Numerical Analysis, 28*(4), 1165-1182.

Mechee, M., Senu, N., Ismail, F., Nikouravan, B. & Siri, Z. (2013). A three-stage fifth-order Runge-Kutta method for directly solving special third-order differential equation with application to thin film flow problem. *Mathematical Problems in Engineering.*

Mechee, M.S., Kadhim, M.A. (2016). Direct explicit integrators of RK type for solving special fourth-order ordinary differential equations with an application. *Global Journal of Pure and Applied Mathematics, 12*(6), 4687-4715.

Mechee, M.S., Mshachal, J.K. (2019). Derivation of direct explicit integrators of RK type for solving class of seventh-order ordinary differential equations. *Karbala International Journal of Modern Science, 5*(3), 8.

Mechee, M.S., Hussain, Z.M. & Mohammed, H.R. (2016). On the reliability and stability of direct explicit Runge-Kutta integrators. *Global Journal of Pure and Applied Mathematics, 12*(4), 3959-3975.

Mechee, M.S., Kadhim, M.A. (2016). Explicit direct integrators of RK type for solving special fifth-order ordinary differential equations. *American Journal of Applied Sciences, 13*(19), 1452-1460.

Mechee, M.S. (2019). Generalized RK integrators for solving class of sixth-order ordinary differential equations. *Journal of Interdisciplinary Mathematics, 22*(8), 1457-1461.

Ray, S.S., Gupta, A.K. (2017). Two-dimensional Legendre wavelet method for travelling wave solutions of time-fractional generalized seventh order KdV equation. *Computers & Mathematics with Applications, 73*(6), 1118-1133.

Reddy, Y.R. (2017). Efficient numerical solution for seventh order differential equation by using septic B-spline collocation method with non-uniform length. *Int. J. of Pure and Applied Mathematics, 13*(7), 3543-3548.

A.M. Wazwaz 2002 Wazwaz, A.M. (2002). A new method for solving singular initial value problems in the second-order ordinary differential equations. *Applied Mathematics and Computation, 128*(1), 45-57.